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HEAT TRANSFER THROUGH THE TURBULENT INCOMPRESSIBLE BOUNDARY LAYER IN THE PRESENCE OF MODERATE PRESSURE GRADIENT

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Abstract—From the assumption that the "universal law of the wall" is applicable to the turbulent boundary layer for moderately accelerating and decelerating flows along a wall it is shown that the thickness and the eddy diffusivity variation through the thickness can be derived at any Reynolds number.

With the additional assumption that the eddy diffusivities for momentum and heat are equal solutions were carried out to the energy equation to obtain Stanton number variations with Reynolds number for both uniform wall temperature and uniform wall heat flux. Two Reynolds number values were considered at which the heating commenced, Prandtl numbers of 0·01, 0·7 and 10 were used and these cases were examined for a number of arbitrarily chosen uniform pressure gradient parameters corresponding to one dimensional diverging or converging ducts.

NOMENCLATURE

A, cross-sectional area of a converging or diverging passage;
A_i, area of passage at $x = 0$;
C_f, local coefficient of friction $(= \tau_w/\frac{1}{2}\rho u_s^2)$;
C_p, specific heat at constant pressure;
h, local heat-transfer coefficient;
l, unheated starting length;
Pr, Prandtl number, ν/α;
q_w, wall heat flux;
R, Reynolds number defined as $\int_0^x (u_s\, dx/\nu)$;
R_l, Reynolds number at the position l where the step in temperature or heat flux occurs;
St, local Stanton number;
t, temperature;
t_w, wall temperature;
t_s, free stream temperature (constant);
u, velocity in the x direction;
u_i, initial velocity at $x = 0$;
u_s, free stream velocity (a function of x);
u^+, dimensionless velocity, $u/\sqrt{(\tau_w/\rho)}$;
v, velocity component in the y direction;
x, distance from the leading edge;
y, distance normal to the wall;
y^+, dimensionless distance normal to the wall;
y_s^+, y^+ at the edge of the boundary layer where $u^+ = u_s^+$;
Z, Pressure gradient parameter, $(1/u_s)\,(du_s/dR)$;
α, thermal molecular diffusivity, $k/\rho C_p$;
β, $(1/Pr) + (\epsilon_h/\nu)$;
ΔR, step length in R;
Δy^+, step length in y^+;
ϵ_h, eddy diffusivity for heat;
ϵ_m, eddy diffusivity for momentum;
θ, dimensionless temperature $(t - t_w)/(t_s - t_w)$;
θ', dimensionless temperature, $q_w/\rho C_p u_i$;
ν, kinematic viscosity;
ρ, density;
τ, shear stress;
τ_w, shear stress at the wall;
τ, stream function.

INTRODUCTION

THIS ARTICLE is an extension to an earlier one [1] in which a solution was presented to the problem of the heat transfer from a plate at uniform temperature, except for an initial length at the

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stream temperature, to a fluid stream moving at constant velocity. The turbulent boundary layer was assumed to be established from the leading edge.

It has since been realised that the same procedure can be applied to accelerating and decelerating flows if the same basic assumptions are postulated. These assumptions are:

1. The fluid physical properties are constant, i.e. solutions are valid for only moderate temperature differences and velocities.

2. The "universal law of the wall" applies throughout the whole thickness of the boundary layer. The model taken for the layer is that the same relation between u^+ and y^+ applies at any value of R out to the value y_s^+ when the velocity u_s^+ is reached. Beyond this value of y_s^+ the dimensionless velocity u_s^+ is constant.

However, as R increases both y_s^+ and u_s^+ increase.

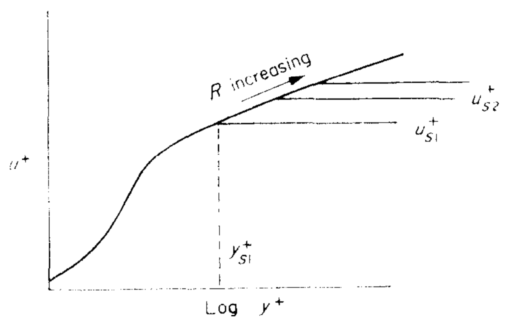

FIG. 1. Assumed model of boundary layer.

3. The eddy diffusivities for momentum and heat are equal.

The second assumption is certainly not valid for flows with large pressure gradients such as in a wide angle diffuser where breakaway soon occurs or in a rapidly converging passage where the boundary layer quickly returns to laminar flow. With moderate pressure gradient however there is some evidence [2–6] that this approximation is reasonable. Such cases occur in practice with flows in the hydrodynamic entrance region of ducts, between adjacent fins of a finned tube and over bluff bodies such as cones or wedges.

OUTLINE OF PROCEDURE

Under the above assumptions it will be shown that relations can be obtained, through the momentum equation, which yield the variation of thickness y_s^+ of the boundary layer with Reynolds number, R. Also at any given position the same analysis will give the variation through the boundary layer of the eddy diffusivity for momentum.

These relations enable a numerical solution to be undertaken of the energy equation of the boundary layer which gives Stanton number–Reynolds number variations for different thermal boundary conditions.

MOMENTUM EQUATION

The procedure to be followed in transforming the momentum equation into the variables u^+ and y^+ is very similar to that given in [1]. In this extension however the free stream velocity u_s is not constant but is a function of x only.

The well-known form of the momentum equation for the boundary layer is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial \tau}{\partial y} + u_s \cdot \frac{du_s}{dx} \tag{1}$$

A stream function ψ is chosen to satisfy the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

and

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

Using the von Mises transformation yields,

$$u\frac{\partial u}{\partial x} = \frac{u}{\rho}\frac{\partial \tau}{\partial \psi} + u_s\frac{du_s}{dx} \tag{3}$$

Introduction of the dimensionless variables u^+ and y^+ bearing in mind that τ_w is a function of x only yields

$$\frac{\partial \psi}{\partial y^+} = \nu u^+$$

The assumption that u^+ is a function of y^+ only enables this to be written

$$\frac{d\psi}{dy^+} = \nu u^+$$

and substitution for ψ is now possible in (3) to obtain,

$$u\frac{\partial u}{\partial x} = \frac{u}{\rho \nu u^+}\frac{\partial \tau}{\partial y^+} + u_s \frac{du_s}{dx} \tag{4}$$

The Reynolds number is defined in the form

$$R = \int_0^x \frac{u_s\, dx}{\nu}$$

Since R is a function of x alone and using the additional substitution

$$u = \frac{u^+ u_s}{u_s^+}$$

then (4) becomes

$$\frac{u^+}{u_s^+}\frac{\partial}{\partial R}\left(\frac{u^+ u_s}{u_s^+}\right) = \frac{1}{\rho\, u_s^+\, u_s}\frac{\partial \tau}{\partial y^+} + \frac{du_s}{dR}$$

Expanding the L.H.S.

$$\frac{u^+}{u_s^+}\left[\frac{u_s}{u_s^+}\frac{\partial u^+}{\partial R} - \frac{u_s\, u^+}{(u_s^+)^2}\frac{\partial u_s^+}{\partial R} + \frac{u^+}{u_s^+}\frac{du_s}{dR}\right] = \frac{1}{\rho\, u_s^+\, u_s}\frac{\partial \tau}{\partial y^+} + \frac{du_s}{dR}$$

The assumption of $u^+ = f(y^+)$ only means $(\partial u^+/\partial R) = 0$. The above may now be simplified to yield

$$-(u^+)^2\frac{du_s^+}{dR} = \frac{1}{\tau_w}\frac{\partial \tau}{\partial y^+} + [(u_s^+)^3 - u_s^+(u^+)^2]\frac{1}{u_s}\frac{du_s}{dR} \tag{5}$$

Now at the wall $y^+ = 0$, $\tau = \tau_w$ and the above may be integrated to any position y^+

$$-\frac{du_s^+}{dR}\int_0^{y^+}(u^+)^2\, dy^+ = \frac{\tau - \tau_w}{\tau_w} + Z\left[(u_s^+)^3 y^+ - u_s^+\int_0^{y^+}(u^+)^2\, dy^+\right]$$

where Z is the pressure gradient parameter

$$\frac{1}{u_s}\frac{du_s}{dR}$$

Re-arranging

$$\frac{\tau}{\tau_w} = 1 - \frac{du_s^+}{dR}\int_0^{y^+}(u^+)^2\, dy^+ - Z\left[(u_s^+)^3 y^+ - u_s^+\int_0^{y^+}(u^+)^2\, dy^+\right] \tag{6}$$

When

$$y = y_s^+ \text{ then } \tau = 0$$

Hence

$$\frac{du_s^+}{dR}\int_0^{y_s^+}(u^+)^2\, dy^+ = 1 - Z\left[(u_s^+)^3 y_s^+ - u_s^+\int_0^{y_s^+}(u^+)^2\, dy^+\right] \tag{7}$$

Substituting

$$\frac{\tau}{\tau_w} = 1 - \left\{1 + Z\left[u_s^+\int_0^{y_s^+}(u^+)^2\, dy^+ - (u_s^+)^3 y_s^+\right]\right\}\frac{\int_0^{y^+}(u^+)^2\, dy^+}{\int_0^{y_s^+}(u^+)^2\, dy^+} - Z\left[(u_s^+)^3 y^+ - u_s^+\int_0^{y^+}(u^+)^2\, dy^+\right] \tag{8}$$

Equation (7) essentially is an ordinary differential equation which will yield a relation between the dimensionless thickness y_s^+ and Reynolds number. Equation (8) for any chosen y_s^+ will yield the variation of shear stress ratio.

The eddy diffusivity for momentum variation follows from the shear stress ratio since

$$\frac{\epsilon_m}{\nu} = \frac{\tau}{\tau_w}\frac{dy^+}{du^+} - 1 \tag{9}$$

Equation (7) may be rewritten

$$\frac{dR}{dy_s^+} = \frac{\int_0^{y_s^+}(u^+)^2\, dy^+}{\left\{1 - \left[(u_s^+)^3 y_s^+ - u_s^+\int_0^{y_s^+}(u^+)^2\, dy^+\right] Z\right\}\frac{dy_s^+}{du_s^+}} \tag{10}$$

Since

$$\frac{du_s^+}{dR} = \frac{du_s^+}{dy_s^+} \cdot \frac{dy_s^+}{dR}$$

The above equations are fairly general and before proceeding to particular solutions it is worth pointing out two features:

(a) It should be noticed that they apply to any form of the u^+–y^+ relation provided this does not change along the R direction. It is known that the relation obeys the well-known logarithmic law near to the wall but departs from this in the outer part—sometimes referred to as the wake. In this analysis the simple assumption will be made that the logarithmic form is obeyed for $y^+ > 26$ but an extension to a different law for the outer part could be carried out.

In the analysis Deissler's form of the law of the wall is used namely,

$$0 < y^+ < 26 \quad \frac{du^+}{dy^+} = \frac{1}{1 + 0{\cdot}0154\, u^+y^+ [1 - \exp(-0{\cdot}0154\, u^+y^+)]}$$

$$26 < y^+ < y_s^+ \quad u^+ = \frac{1}{0{\cdot}36} \ln\left(\frac{y^+}{26}\right) + 12{\cdot}8426$$

(b) The pressure gradient parameter

$$Z = \frac{1}{u_s} \frac{du_s}{dR}$$

will in general be a function of the Reynolds number but in this analysis only constant values of the parameter have been considered since the analysis of this case is much simpler. For external flows, such as flow over a wedge, the parameter variation may be obtained from the potential flow and therefore quite independently of the boundary layer analysis. For internal flows, however, there will be coupling between the boundary layer analysis and the free stream variation. For example, in the entrance region of a duct with parallel walls the pressure drop is itself caused by the boundary layer growth. To include such coupling will complicate the analysis but it should still be possible through the same basic approach.

If it be assumed that the boundary layers are thin, then constant values of the pressure gradient parameter will yield solutions to straight sided converging and diverging passages. The growth of the boundary layer in practice, however, implies that such solutions will only approximate to these cases, perhaps better for convergences, and these solutions will, in fact, apply to passages with walls with some slight, but unknown, curvature.

To apply the general solutions to be obtained to the simplified cases, useful expressions are.

$$\frac{1}{u_s} \frac{du_s}{dR} = -\frac{\nu}{c} \frac{dA}{dx} \tag{11}$$

where $c = u_s A$ (constant)
and

$$R = -\frac{c}{\nu(dA/dx)} \ln\left(1 + \frac{x}{A_i} \frac{dA}{dx}\right) \tag{12}$$

where A_i is the initial area of the duct.

FRICTION FACTOR, SHEAR STRESS AND EDDY DIFFUSIVITY VARIATIONS

It is unrealistic to assume that the growth of the boundary layer follows the universal law in the early stages near the leading stage. Nevertheless, previous analyses in which the universal law was assumed from the leading edge have always shown good agreement with experimental friction factors. Accordingly this assumption has been used in this analysis and, although equation (10) was solved for values of y_s^+ greater than 26, a correction was made to the Reynolds numbers for the growth out to $y_s^+ = 26$. This correction was taken to be the same as for no pressure gradient namely $R_{y_s^+ = 26} = 4759$ (see [1] for details).

Figure 2 shows solutions to equation (10) namely, the R–y_s^+ relation for various constant values of the pressure gradient parameter. Note that the pressure gradient does not affect the y_s^+ values near the leading edge. The figure also shows the rapid growth of y_s^+ for diverging flows (Z negative) and the converse for converging flows.

Figure 3 shows shear stress ratios for an arbitrary selection of pressure gradient parameters and at a particular value of y_s^+. An important feature of the positive pressure gradient

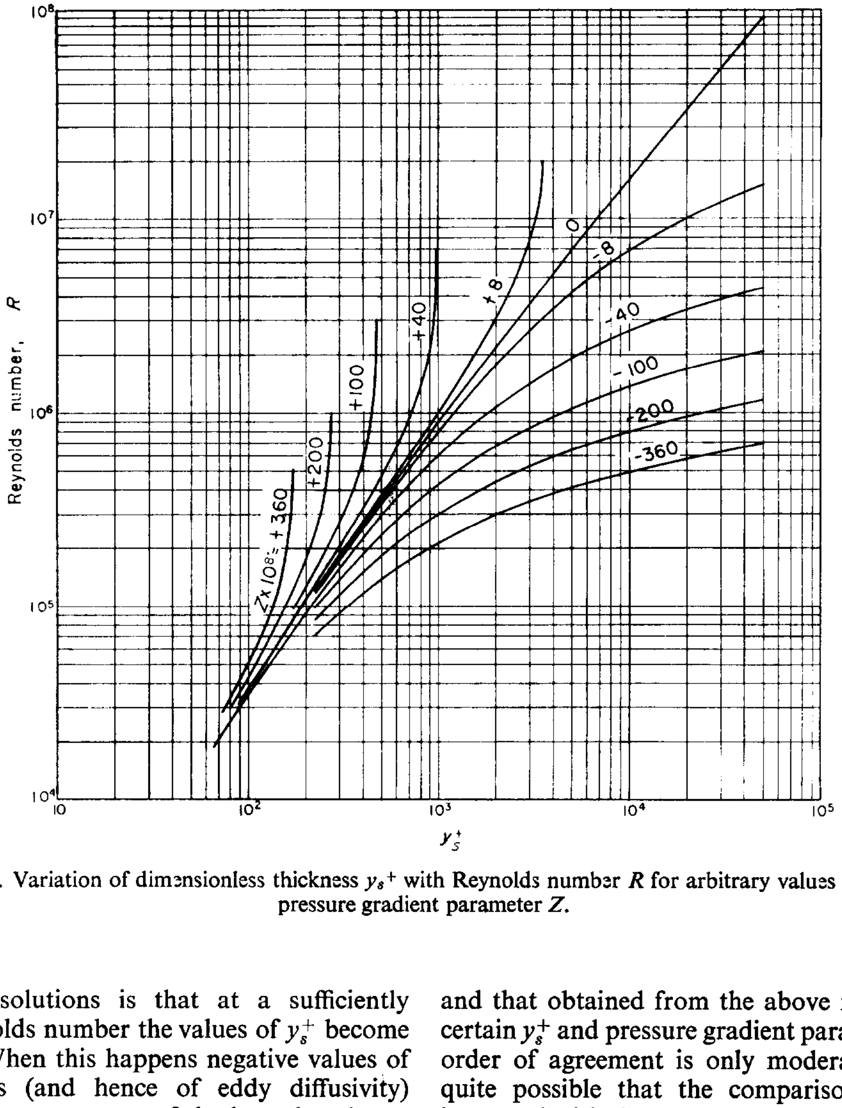

FIG. 2. Variation of dimensionless thickness y_s^+ with Reynolds number R for arbitrary values of the pressure gradient parameter Z.

parameter solutions is that at a sufficiently large Reynolds number the values of y_s^+ become constant. When this happens negative values of shear stress (and hence of eddy diffusivity) appear in the outer part of the boundary layer. This, of course, is impossible and solutions were discontinued beyond this point. Perhaps this point has some relation to that of "reverse transition" described by Kays and Moretti [7] at which the boundary layer returns to laminar flow.

Schubauer and Klebanoff [8] obtained variations of shear stress through the turbulent boundary layer. It is not possible to infer the value of y_s^+ to which their results apply but Fig. 4 gives a comparison between their experimental results and that obtained from the above relations for certain y_s^+ and pressure gradient parameters. The order of agreement is only moderate but it is quite possible that the comparison could be improved with the correct values of the pressure gradient parameter and y_s^+. Brand and Persen [9] have also calculated shear stress variations using very similar methods to those given here but using Spalding's form of the law of the wall [10] and these agree well.

Local friction factors are plotted on Fig. 5 and typical eddy diffusivity variations on Fig. 6. The latter curves form the main object of this section of the analysis and, together with the y_s^+–R variation, permit a solution of the energy equation to be attempted.

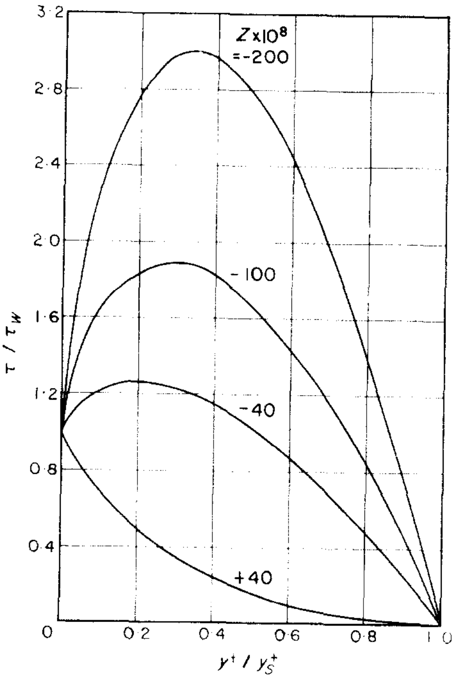

FIG. 3. Distribution of shear stress through the boundary layer for different values of the pressure gradient parameter Z ($y_s^+ = 1000$).

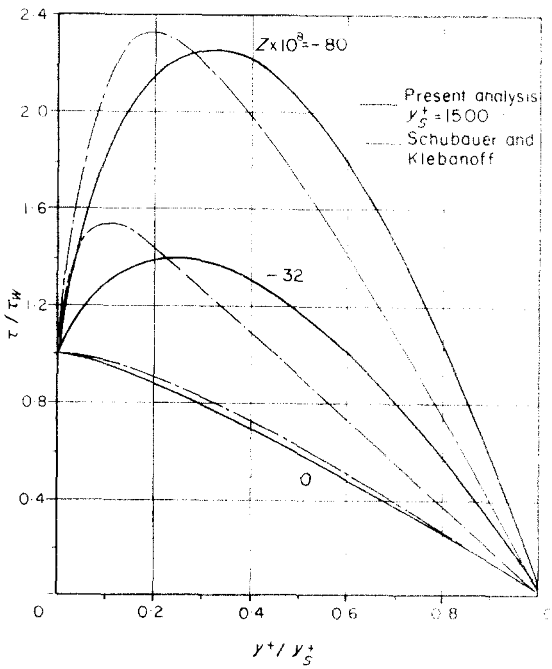

FIG. 4. Comparison between predicted shear stress variations and the experiments of Schubauer and Klebanoff [8].

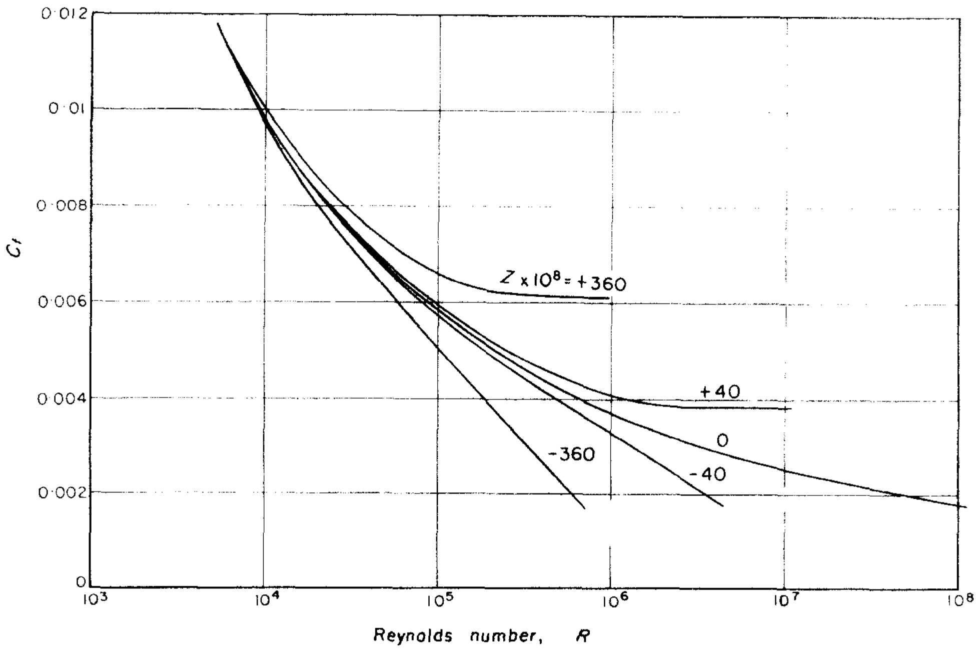

FIG. 5. Friction factor variations for different values of the pressure gradient parameter Z.

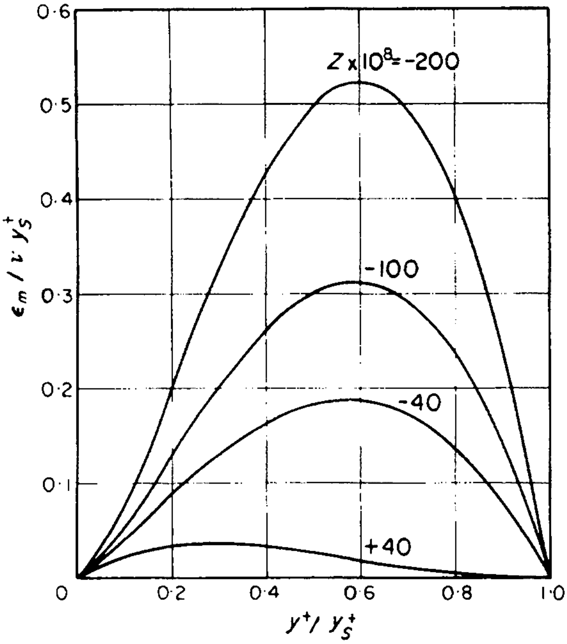

FIG. 6. Distribution of eddy diffusivity through the boundary layer for different values of the pressure gradient parameter Z. ($y_s^+ = 1000$).

ENERGY EQUATION SOLUTION

The form of the energy equation, under the assumptions of constant physical properties and negligible dissipative effects, is the same as for zero pressure gradient, namely,

$$u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = \frac{\partial}{\partial y}\left[(a + \epsilon_h)\frac{\partial \theta}{\partial y}\right]$$

or, using the same transformations as in the momentum equation,

$$u^+ u_s^+ \frac{\partial \theta}{R} = \frac{\partial}{\partial y^+}\left\{\left(\frac{1}{Pr} + \frac{\epsilon_h}{\nu}\right)\frac{\partial \theta}{\partial y^+}\right\}$$

Subject to the boundary conditions discussed below the equation was solved by a finite difference technique which is described in detail in [1]. The computer programme was arranged to adjust the step length in Reynolds number to conform to stability limitations.

Because the programme to solve the energy equation involves the momentum solution it would have been too wasteful of computer time to solve equation (10) every time a different Prandtl number or starting length was calculated. Hence, sixth degree polynominal approximations were used for the R–y_s^+ relation shown on Fig. 2. The eddy diffusivity variations were calculated in the programme through equations (8) and (9). An initial choice of the increments of Δy^+ was made and when the thermal boundary layer had grown to span the initial range of y_s^+ arrangements were made in the programme to increase the value of Δy^+. The method used was to double the increment and discard intermediate values of the dimensionless temperature which produced a slight disturbance in the results.

UNIFORM WALL TEMPERATURE

The boundary condition here is

$$\theta = 1 \quad \text{all } y^+ \quad x < l$$

$$\theta = 0 \quad \text{at } y^+ = 0 \quad x > l$$

The problem was solved in two parts: (a) a solution using small steps of Δy^+ which was rather slow in increasing R; (b) a solution in which a steady state was assumed in the layer up to $y^+ = 26$.

A plot of results showed that solution (a) soon merged with that of (b) and beyond this point the solution (b) was used.

UNIFORM WALL HEAT FLUX

In this case it is necessary to define a new dimensionless temperature θ' where

$$\theta' = \frac{t - t_s}{q_w / \rho\, c_p\, u_i}$$

u_i is the initial free stream velocity where $R = 0$.

The energy equation takes the same form as above with this dimensionless variable.

The boundary condition of uniform heat flux must be introduced into the numerical method and this was achieved as follows:

$$\frac{q_w}{\rho\, c_p\, u_i} = - \frac{a + \epsilon_h}{u_i}\left(\frac{\partial t}{\partial y}\right)_{y=0}$$

$$\therefore \left(\beta \frac{\partial \theta'}{\partial y^+}\right)_{y^+=0} = - u_s^+ \frac{u_i}{u_s}$$

where

$$\beta = \frac{1}{Pr} + \frac{\epsilon_h}{\nu}$$

Now the pressure gradient parameter

$$Z = \frac{1}{u_s}\frac{\mathrm{d}u_s}{\mathrm{d}R}$$

$$u_s = u_i \quad \text{at} \quad R = 0$$

$$\therefore R = \frac{1}{Z}\ln\frac{u_s}{u_i}$$

or

$$\frac{u_s}{u_i} = \exp(ZR)$$

$$\left(\beta\frac{\partial\theta'}{\partial y^+}\right)_{y^+=0} = -\frac{u_s^+}{\exp(ZR)}$$

The quantity on L.H.S. above appears in the numerical method and is used to infer the next wall temperature (this is, in fact, the constant c of reference 1).

The calculation was again carried out in two parts: (a) a solution near the nose using small steps of $\Delta y^+ = 2$ out from the wall; (b) again assuming a steady state solution for the layer up to $y^+ = 26$.

In fact, it was found that the solution near to the step was hardly affected by the pressure gradient and only a few calculations of this type were made.

For the assumption (b) we obtain

$$\theta'_w = \theta'_{26} + \frac{u_s^+}{\exp(ZR)}\int_0^{26}\frac{1}{\beta}\,\mathrm{d}y^+.$$

Values of

$$\int_0^{26}\frac{1}{\beta}\,\mathrm{d}y^+$$

are given in [1] for different Prandtl numbers.

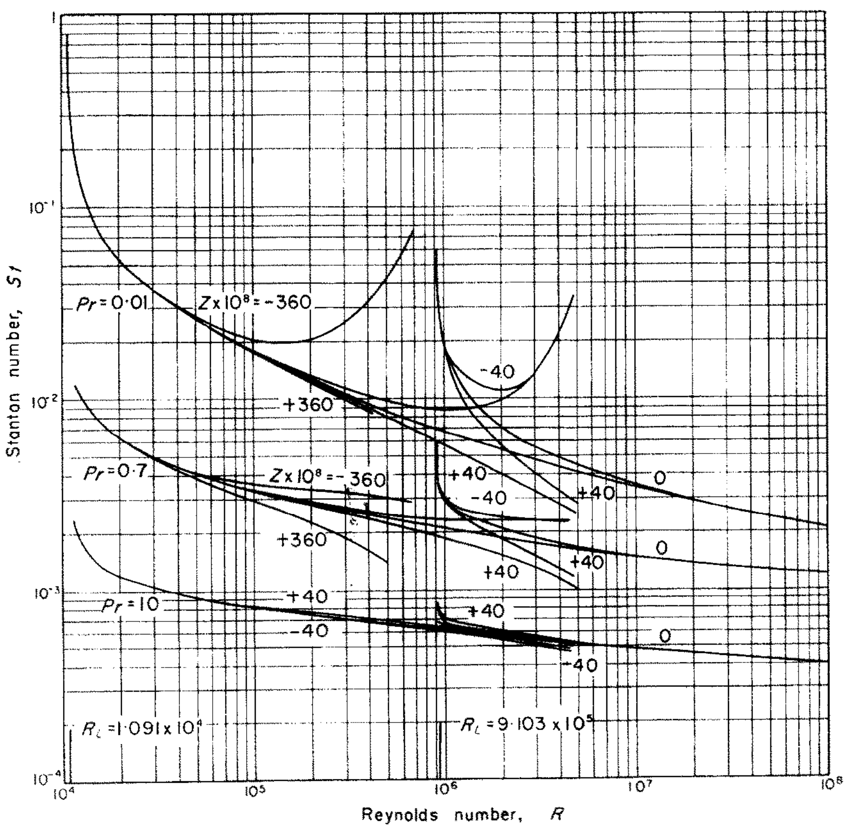

FIG. 7. Stanton numbers for different pressure gradients with a step in uniform surface temperature at two values of R_i.

The Stanton number follows from

$$St = \frac{1}{\theta'_w} \cdot \frac{u_i}{u_s} = \frac{1}{\theta'_{26} \exp(ZR) + u_s^+ \int_0^{26} (1/\beta)\, dy^+}.$$

RESULTS

The calculations outlined above were carried out on the Manchester University Atlas Computer and the results in the form of Stanton number versus Reynolds number or y_s^+ are shown on Figs. 7, 8, 9 for different Prandtl numbers, pressure gradient parameters and unheated starting lengths.

Only two starting lengths are considered together with three Prandtl numbers and several positive and negative pressure gradient parameters. A positive value of the parameter corresponds to a converging passage. These calculations are very expensive in computer time since the stability limitation permits only a slow increase in Reynolds number and this factor reduced the number of combinations which could be examined. However, the calculation was repeated for the same parameters for both uniform wall heat flux and uniform wall temperature and the close similarity between the two was interpreted as convincing check of the numerical procedures.

The results for uniform heat flux lie above those for uniform wall temperature but only at $Pr = 0{\cdot}01$ is the difference significant.

The influence of the presence of accelerating and decelerating flows is clearly significant. On the St–R plots the accelerating flows lie below and the decelerating flows above, the zero

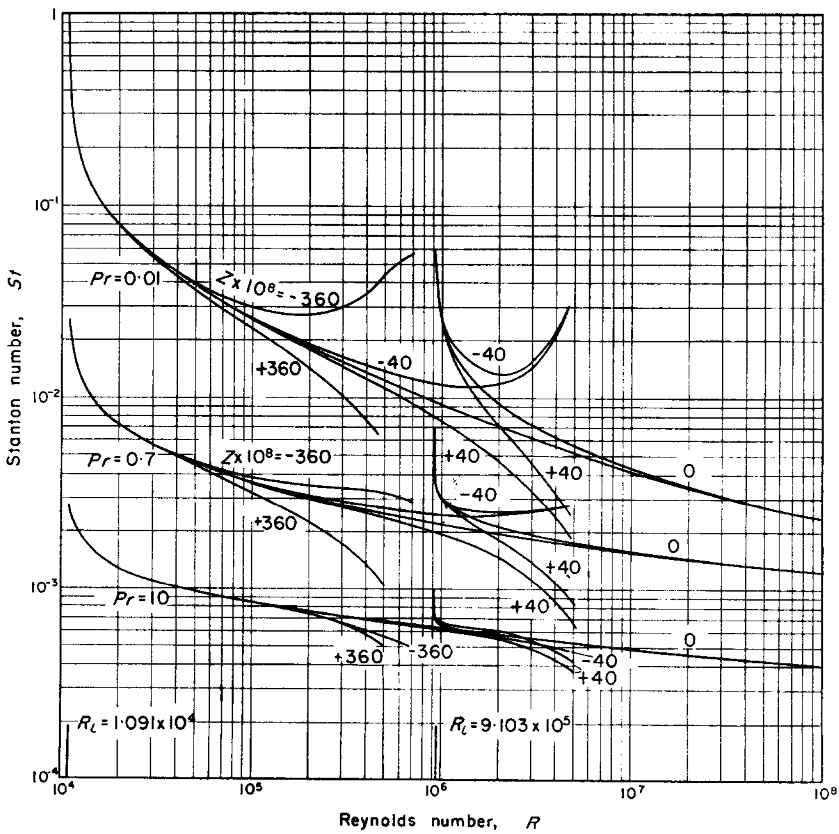

FIG. 8. Stanton numbers for different pressure gradients with a step in uniform heat flux at two values of R_l.

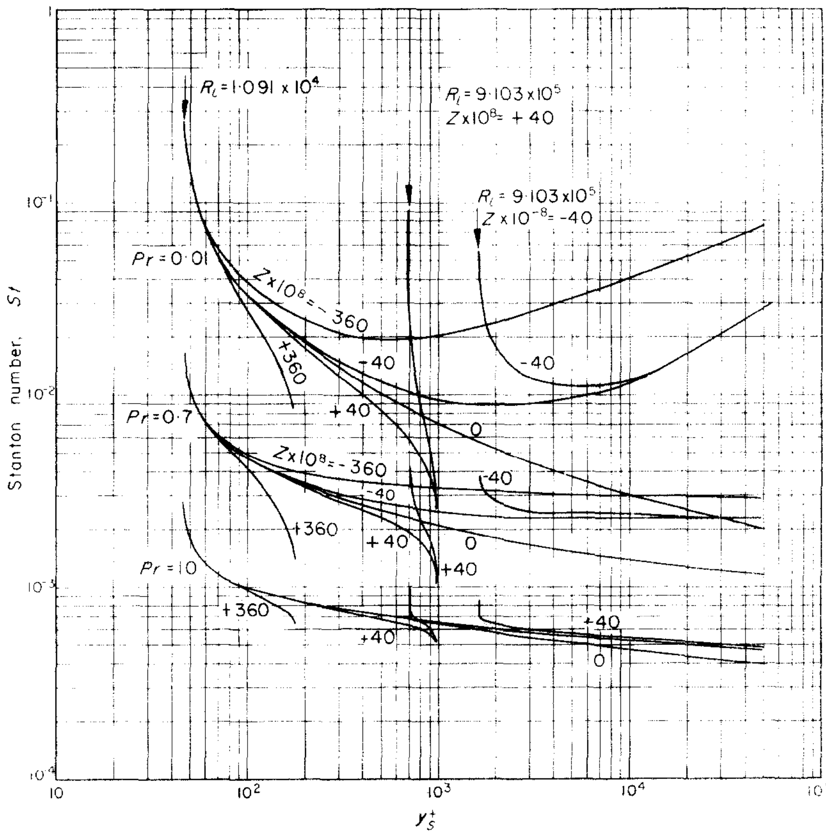

FIG. 9. Stanton number variation with y_s^+ for different pressure gradient parameters (step in temperature).

pressure gradient curves. Again the effects are more significant at low Prandtl number. A feature of the negative pressure gradients is that the Stanton number curve may start to rise after some increase of Reynolds number. This is presumably due to the very high values of eddy diffusivity which may be generated in such boundary layers (see Fig. 6) which offsets the opposite effect due to the growth in thickness of the boundary layer.

It should also be noted that it is not possible to carry calculations to such a high value of R as for no pressure gradient. For the positive parameters a point is soon reached where y_s^+ becomes constant and here negative values of τ and hence of ϵ_m appear in the outer part of the boundary layer (see Figs. 2, 3) and the programme was automatically halted at this point. Again for the negative parameters the growth in y_s^+ is very rapid and it was only possible to take this parameter to 5×10^4—the limit of Fig. 2.

However, curves such as Fig. 7 or 8 are not strictly comparable as regards the influence of different pressure gradients. Obviously the flows in a converging and diverging duct are not dynamically similar and such simple comparisons are deceptive. For example, for the same inlet velocity and area, then a given Reynolds number corresponds to a much shorter distance in a converging duct than a diverging duct.

A numerical example may illustrate this point. Consider a straight sided duct 4 ft long, end areas 1 ft^2 and 0·6 ft^2. Air ($\rho = 0{\cdot}0807$ lb/ft^3, $C_p = 0{\cdot}24$) flows through the duct and the velocity at the 1 ft^2 section is 40 ft/s. The pressure

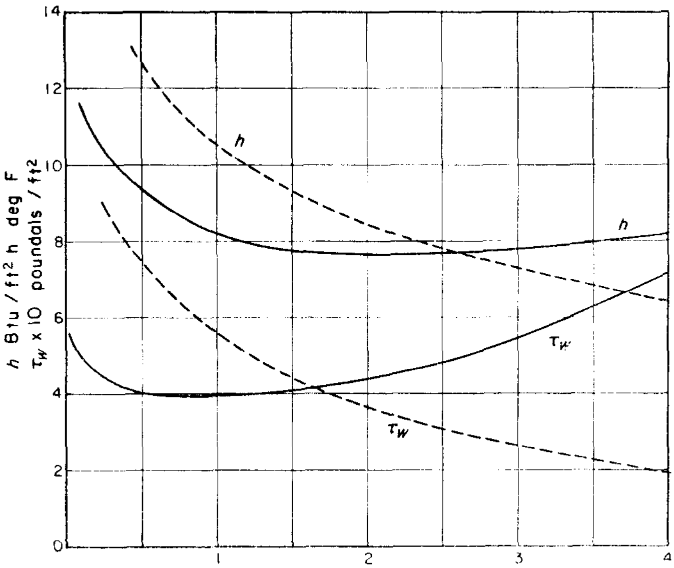

FIG. 10. Variation of heat-transfer coefficient and wall shear stress along a converging and a diverging passage—see example in text. (——— convergence, - - - - - - divergence).

gradient parameter corresponding to this situation is $\pm 40 \times 10^{-8}$ depending on whether the passage is treated as a convergence or a divergence. Using equations (11) and (12) and Fig. 7 for a uniform wall temperature then Fig. 10 may be derived. This shows the local heat-transfer coefficient and wall shear stress variation along the length of the passage. For this example the effect of the pressure gradient on the heat-transfer coefficient is not large and if the no pressure gradient curve of Fig. 7. had been used then the maximum error would be about -10% for divergence and about $+12\%$ for convergence.

Experiments are being conducted to test the analysis and it is hoped to report these in the near future.

CONCLUSION

This analysis has shown that the use of the universal form of the law of the wall will yield a very complete description of the properties of turbulent boundary layers. It is probable that these properties, such as eddy diffusivity are very sensitive to small changes in the form of the law of the wall. Certainly, the results obtained in this article are only valid for equilibrium boundary layers where the pressure gradients are relatively small. Experiments are required to discover the limits to which the assumptions may be carried but it is suggested that, within these limitations, the analysis provides a useful method of solution to a wide range of practical problems.

ACKNOWLEDGEMENT

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Résumé—On montre, â partir de l'hypothèse que la "loi universelle à la paroi" s'applique à la couche limite turbulente pour des écoulements le long d'une paroi avec de faibles accélérations ou décélérations, que l'épaisseur et la variation de la diffusivité turbulente à travers l'épaisseur peut être obtenue pour n'importe quel nombre de Reynolds.

Avec l'hypothèse supplémentaire que les diffusivités turbulentes pour la quantité de mouvement et la chaleur sont égales, on a trouvé des solutions de l'équation de l'énergie afin d'obtenir les variations du nombre de Stanton avec le nombre de Reynolds pour, soit une température pariétale uniforme, soit un flux de chaleur pariétal uniforme. On a considéré deux valeurs du nombre de Reynolds pour le début du chauffage, on a utilisé des nombres de Prandtl de 0,01, 0,7 et 10 et trois cas ont été examinés pour certains paramèters de gradients de pression uniforms arbitrairement choisis correspondant à des conduites uni-dimensionnelles divergentes ou convergentes.

Zusammenfassung—Unter der Annahme, dass das "universelle Wandgesetz" auf die turbulente Grenzschicht von geringfügig beschleunigenden und verzögernden Wandströmungen anwendbar ist wird gezeigt, dass die Grenzschichtdicke und die Änderungen der turbulenten Austauschgrösse abhängig vom Wandabstand für jede Reynoldszahl abgeleitet werden kann.

Mit der zusätzilchen Annahme, dass der turbulente Austausch von Impuls and Wärme gleich ist, wurden Lösungen für die Energiegleichungen ausgearbeitet, um Änderungen der Stanton zahl mit der Reynoldszahl sowohl für gleichförmige Wandtemperatur als auch für gleichformige Wärmestromdichte an der Wand zu erhalten: Für die Reynoldszahl an der Stelle des Heizbeginns wurden zwei Werte angenommen und die Prandtlzahlen 0,01, 0,7 und 10 wurden zugrundegelegt. Diese Fälle wurden untersucht für eine Anzahl beliebig ausgewählter, gleichförmiger Druckgradienten, für eindimensionale divergierende oder konvergierende Strömungskanäle.

Аннотация—Исходя из допущения о применимости «универсального закона стенки» к турбулентному пограничному слою при умеренном увеличении или уменьшении скорости течения вдоль стенки, показано, что величину изменения толщины и коэффициента турбулентной диффузии по толщине можно определтить при любых значениях критерия Рейнольдса.

При дополнительно допущениим о равенстве турбулентного переноса количества движения и тепла проведены решения уравнения энергии для определения изменений критерия Стантона с изменением критерия Рейнольдса как для случая постоянной температуры стенки, так и для случая постоянного теплового потока на стенке.

Рассмотрены два значения критерия Рейнольдса, при которых начинался нагрев. Принимались значения критерия Прандтля 0,01 ; 0,7 и 10, и рассматривался ряд произвольно выбранных параметров градиента однородного давления, соответствующих одномерным расширяющимся или сужающимся каналам.